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DETERMINATION OF THE SHAPE OF A FREE BALLOON

Theoretical Development

Scientific Report No. 1

Prepared for:

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

Contract No. AF 19(628)-2783

Project No. 6665
Task 666501

Report Number 2421

Justin H. Smalley

Electronics Division
GENERAL MILLS, INC.
2295 Walnut Street
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ABSTRACT

This report, the first of a series, summarizes a survey of the literature on thin, flexible, pressure vessels. Equations for the stresses in an axi-symmetric balloon are determined. The equations are then rearranged to be most suitable for computation of shape. Derivation of the balloon design parameter Σ is presented.

I. INTRODUCTION

This report is the first of a series of reports that will examine the problem of determining the shape of free balloons. The designs used today were determined by the University of Minnesota and presented in Volume IX of the reports of their work (Reference 1). This work has withstood the test of time; its correctness has been proved by thousands of sucessful balloon flights. At the time the work was done (1953), the largest balloon designs were about 3 million cubic feet, and maximum payloads were approximately 400 pounds. Since then, balloons and payloads have been flown which fall outside the range of design values provided by the University of Minnesota. In addition, new materials, balloon configurations, and fabrication techniques have been introduced. It has therefore become imperative to reopen the subject of determination of balloon shape.

The University of Minnesota performed their calculations on an analog computer. The work of this study will be done by digital computer, which will permit us to make more accurate calculations, and will afford an independent check of the earlier work.

This first report presents a review of past work on the subject and a derivation of the equations that define the shape of a free balloon.

II. REVIEW OF PREVIOUS EFFORTS

A. Early Work

Early efforts to determine the shape of flexible containers subject to internal pressure were concerned with airship shapes. The problem was considered two-dimensional. The shape of a cross section normal to the longitudinal centerline was determined. For the two-dimensional case, material stresses in the longitudinal direction could be ignored or assumed equal to zero. While this work is not directly applicable to free-balloon design, we have examined it for methods of approach to the problem.

The earliest study of the general problem is cited in Reference 2 with a date of 1914. The study was a graphical solution that is simple and straightforward, which can be made as accurate as desirable. At any point the stress t is determined simply by the expression

$$t = pr$$

where

$$p = \text{local pressure}, \qquad r = \text{radius of curvature}.$$

For short increments of the curve, the above values are constant, and an arc is struck of radius r. At the end of the arc, new values of p and r are calculated, and the next arc is drawn. Adjoining radii of curvature are colinear. Reference 2 does not state how the solution is started. We surmise that t is determined from considerations of payload and shape and that the desired size and pressure are found by a few trial solutions. It is convenient to start at the top of the balloon where the direction of the radius of curvature is known. A sketch illustrating this method is presented as Figure 1.

In this and most other work the weight of the material is assumed to have a negligible effect. Such an assumption is not far wrong, but neither is it necessary. The material weight changes stress in proportion to the vertical component of the curve. A similar procedure could be applied to a free balloon except that the gore width would be variable. Total load in the balloon gore would be the defining parameter. As the shape of the balloon developed, local radius and local gore width would be found and would allow determination of local stress and radius of curvature. This method would not be practical for balloons with circumferential stress other than zero.

An interesting analytical solution to the same problem of airship shape is given by Pagon in Reference 3. The solution is in terms of elliptic integrals, so he has devised a chart that covers a large class of solutions. They all have a horizontal tangent at the top, and only superpressures are considered. The solutions are for weightless material only, but the case of envelope weight and envelope shear is discussed. These considerations make the problem intractable analytically.

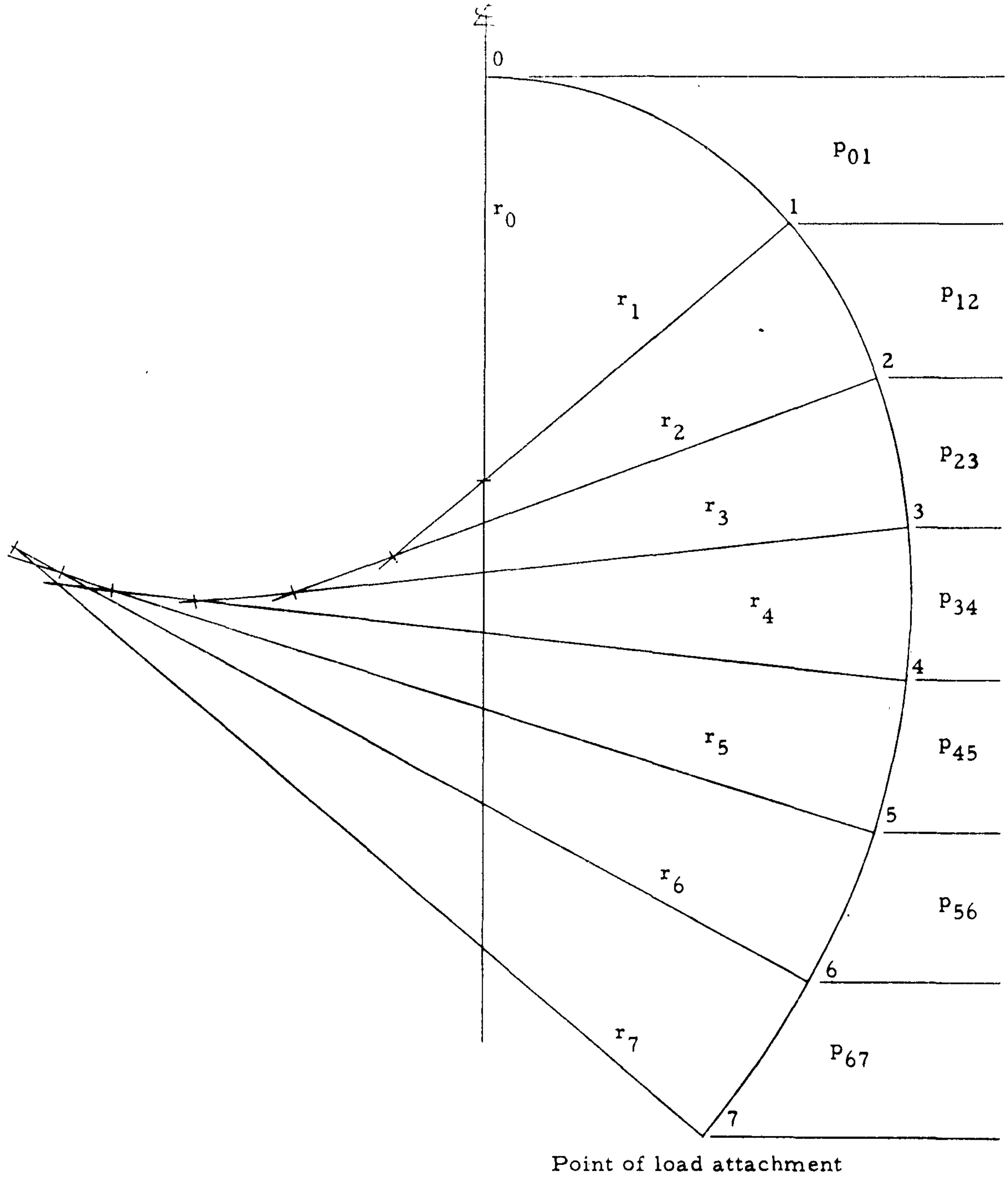


Figure 1. Sketch Illustrating Graphical Determination of Airship Cross Section

The theory of a mechanical method for determining shape is also given in Reference 3. This method was published by a Colonel Crocco in 1914. He showed that the equation for a thin elastic spline loaded at its ends is the same as that for the airship envelope. The procedure for use of such a method is outlined in some detail.

B. Parachute Design

It is appropriate to digress at this point and mention the work that has been done in parachute design. Two assumptions appear to be universal: 1) parachute material is weightless, and 2) the pressure across the parachute is constant. The results of parachute design calculations are, therefore, not the same as those for a balloon, but they do provide a reasonable check on balloon computations. The case of constant pressure is approached in a balloon as the head of gas approaches infinity, or if the balloon is filled with air. Reference 4 states that symmetric structures subject to the above limitations may be represented by

$$\sin \phi = \frac{1}{\lambda} \left[\lambda - 1 + \left(\frac{r}{a}\right)^2 \right]$$

where

ϕ = angle between the curve and a plane normal to the axis of symmetry

λ = shape parameter

r = radial distance from axis of symmetry

a = maximum value of r.

It also states, "By varying the parameter λ a family of curves is generated, identical with the elastica occurring in the theory of the buckling of columns and struts as originally investigated by Euler."

When $\lambda = 1$ the curve is flat-topped and is the so-called Taylor shape. Integration of the above equation and sample curves is given in the reference. Circumferential and meridional stresses are also determined. Near the parachute's top these expressions for stress are applicable to balloons. It is shown that when $\lambda = 1$ the circumferential stress is zero, and that the meridional stress becomes infinite as r approaches zero. It is also shown that with a slightly conical top (and therefore some circumferential stress) the meridional stresses become small. The author further suggests that there should be some circumferential stress to stabilize the parachute's shape (not a problem with balloons) and to aid in full deployment of the gores. He finds that the most efficient shape, from a drag-to-surface-area standpoint, has an axial cord and a re-entrant top. We will investigate the effect of a re-entrant top on balloon area (i. e., weight) in a later report.

In both References 4 and 5, the work of Stevens and Jones (Reference 6) is discussed. Here the canopy cords, corresponding to balloon tapes, carry the entire meridional load. The material is allowed to bow outward between the cords. Stresses and gore patterns are discussed. It has been observed that balloon material bows outward between tapes. This work can be used as a guide toward investigating the stresses in balloons.

C. Free Balloon Design

Upson appears to have been the first to record the concept of a "natural" shape for a free balloon. In Reference 7 he considered the stresses in a partially inflated balloon and recognized the absence of circumferential stress. A version of the differential equation for shape is derived but is applied only to the vehicle's upper half.

A full determination of many weightless natural-shape balloons is made in Reference 8 by Upson and others at the University of Minnesota. They show that at the top the ideal curve is a cubic parabola. Circumferential stress near the top is discussed for several shapes; this approaches infinity for curves of degree between 2 and 3. Material deformation precludes this happening in actual practice. The case of a balloon with material weight is treated briefly.

The work of Upson was carried on by the University of Minnesota and reported in various volumes (Reference 1). It is here that the parameter Σ is introduced. A nomograph is given for determining Σ in terms of the payload, volume, and material weight. Finally, tables of balloon shape are given for $0 \leq \Sigma \leq 0.4$ in steps of 0.05. These have provided the basis for balloon designs used up to the present time.

III. DERIVATION OF THE EQUATIONS

Various derivations are given in the literature cited. The University of Minnesota derived the equations in more than one way. The one presented therein uses the variational principle starting with a statement of the total gravitational potential energy. This method is elegant, but its results are the same as those obtained using simple membrane theory, and it falls back on that theory to evaluate a factor.

A. Assumptions

1) The balloon is assumed to be rotationally symmetric about a vertical axis.

2) Meridional and circumferential stresses are assumed to be constant at all points on the circle lying in a plane normal to the axis of symmetry. This precludes the possibility of shear in the balloon.

3) The densities of the inflation gas and surrounding air are constant.

4) The balloon material is inextensible, thin, and incapable of supporting any bending or compressive loads.

B. Symbols

Symbol	Definition	Dimension
a	pressure head at bottom of balloon	length
b	difference in weight densities of air and inflation gas	force per unit volume
k	constant = $(2\pi)^{-1/3}$	-
p	gas pressure across the balloon material	force per unit area
r	radial coordinate of a point on balloon, measured normal to the axis of symmetry	length
t_c	circumferential stress	force per unit length
t_m	meridional stress	force per unit length
s	gore coordinate of a point on the balloon, measured in the meridional direction from the bottom apex	length
w	unit weight of balloon material	force per unit area
z	height coordinate of a point on balloon, measured parallel to the axis of symmetry from the bottom apex	length
A	area of balloon surface	length squared
B	buoyant force on balloon	force
F	vertical load at top apex of balloon	force
G	gross lift of balloon = bV	force
L	payload suspended at bottom apex of balloon	force
T	total film load = $2\pi r t_m$	force
V	balloon volume	length cubed
W	balloon weight	force

Symbol	Definition
α	non-dimensional a
ζ	non-dimensional z
θ	angle between tangent to the balloon surface and the axis of symmetry, measured in a meridional plane
ρ	non-dimensional r
σ	non-dimensional s
τ	non-dimensional t_c
ψ	angle between two meridional planes
ω	non-dimensional w
Σ	balloon design parameter = $w(2\pi/b^2L)^{1/3}$

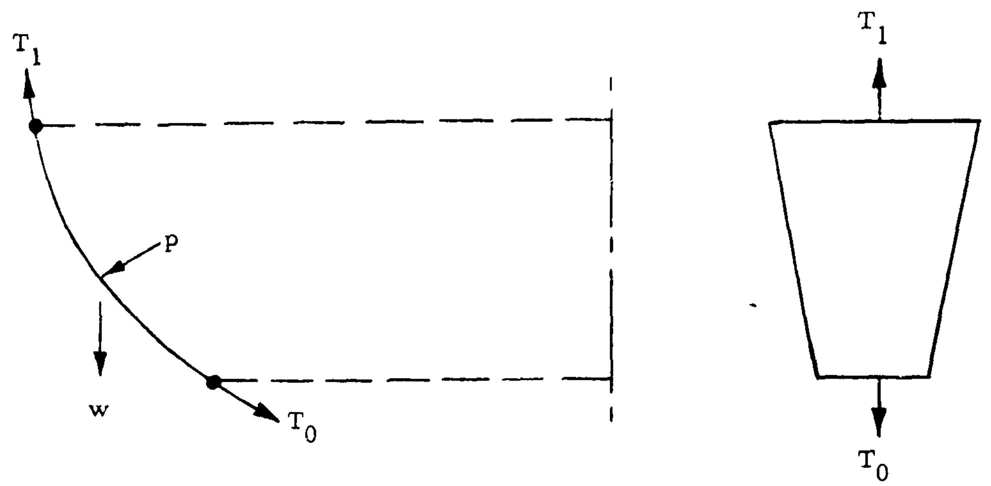
C. Elementary Considerations

At the bottom apex, the payload is supported by the balloon material (film, tapes or both). The material load is easily computed in terms of the apex angle:

$$T = L/\cos\theta_o$$

For film alone, the expression $T = 2\pi r t_m$ is also valid. As r approaches zero, t_m must approach infinity. Thus the payload cannot be supported at a point. Either tapes or sufficient film must be used in an end fitting so that allowable film stresses are not exceeded. This is the basis, of course, for cylinder-end balloons.

Consider a narrow element of material bounded at top and bottom by parallel planes normal to the axis, as sketched:



The forces on the element are shown. The pressure p is everywhere normal to the material. If the material weight is zero, then there is no force or component of force parallel to the element in the meridional direction. The total material loads at the top and bottom of the element must therefore be equal. This may be generalized so far as to say that, for a weightless balloon with zero circumferential stress, the total film load is constant at every level of the balloon. Returning to the sketch, one can see that material weight can be resolved into two components that are normal and parallel to the element. Thus T_1 will be greater than T_0 by the parallel component of the weight. Again, for an actual balloon, the total material load increases with increasing levels by a component of the material weight.

The material load is everywhere finite. It has been shown that in theory the stresses diverge at the bottom. By the same reasoning, for a zero circumferential stress balloon, the stresses will diverge at the top. In a later section, we show how the circumferential stress modifies the meridional stress.

Figure 2 depicts the top portion of a balloon as a free body. Weight and buoyancy are functions of the area and volume of this portion, respectively. When $r \to 0$, then $(W-B) \to 0$, and the force diagram in the figure results. The

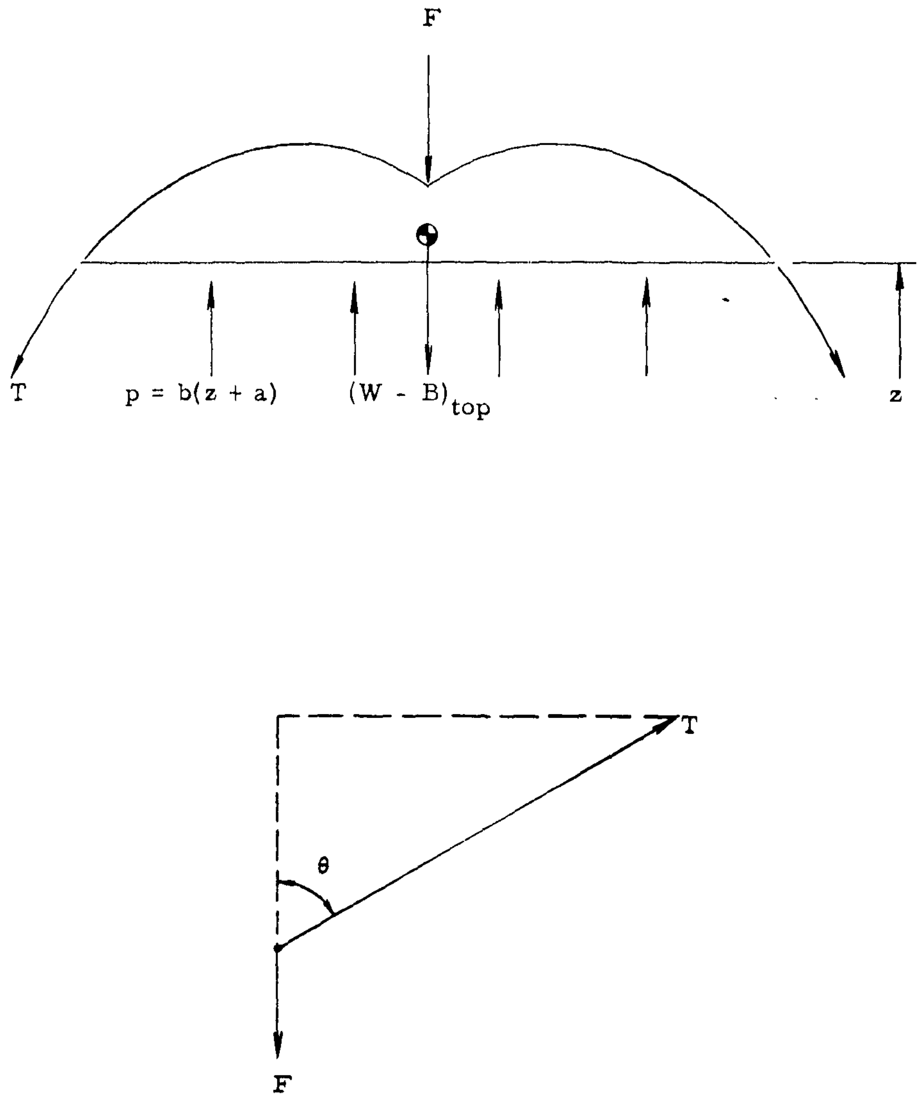


Figure 2. Top Portion of Balloon Considered as a Free Body

material load supports whatever weight F is located at the top. If $F = 0$, then $T \cos \theta = 0$. But T has been shown to be finite, so without a load at the top the balloon must be flat.

D. Derivation of Stress Equations Using Membrane Theory*

Consider an elemental area on the surface of the balloon located as shown in Figure 3. The top and bottom of the element are defined by parallel planes normal to the axis of symmetry and located a distance Δz apart. The intersection of one such plane with the surface of the balloon defines the circumferential direction. The sides of the element are defined by two planes containing the axis of symmetry and separated by an angle ψ ($\psi << 1$). The intersection of one such plane with the surface of the balloon defines the meridional direction. Forces in the meridional and circumferential directions are orthogonal with each other and with the normal to the surface.

The element and the forces acting on it are shown in Figure 4. Summing forces in the vertical plane produces this relationship:

$$(t_m + \Delta t_m)\, c_1 \cos(\theta + \Delta\theta) - t_m c_0 \cos\theta - b(z + a) A \sin\theta - wA = 0$$

where

$$c_0 = r\psi$$

$$c_1 = (r + \Delta r)\psi$$

$$A = 1/2\,(c_0 + c_1)\,\Delta s$$

All infinitesimals of order higher than the first will be omitted. Then $A = r\,\psi\,\Delta s$ and,

*Another derivation, arriving at the same final equations, is given in the Appendix.

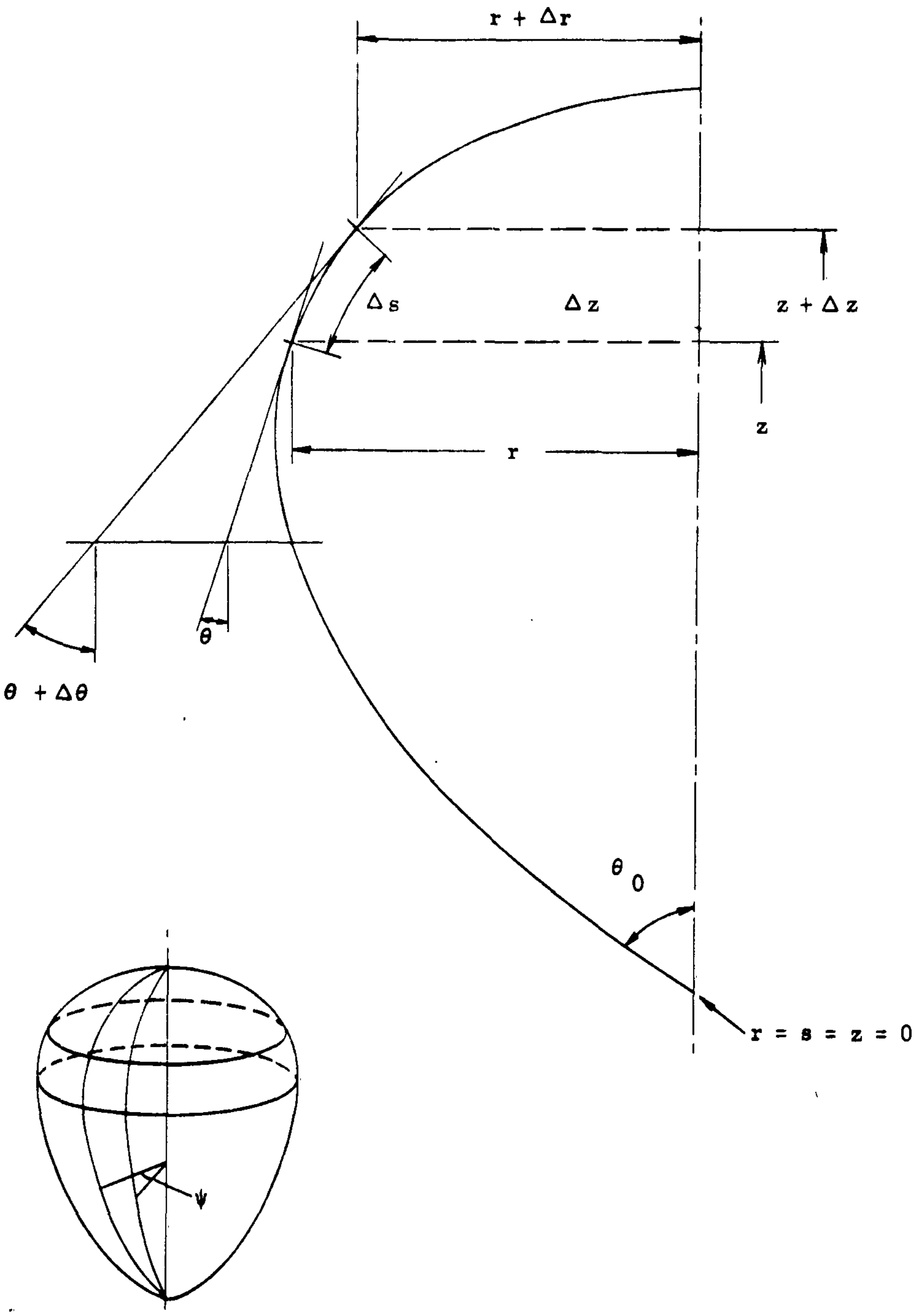


Figure 3. Sketch Showing Relation of Elemental Area and Balloon

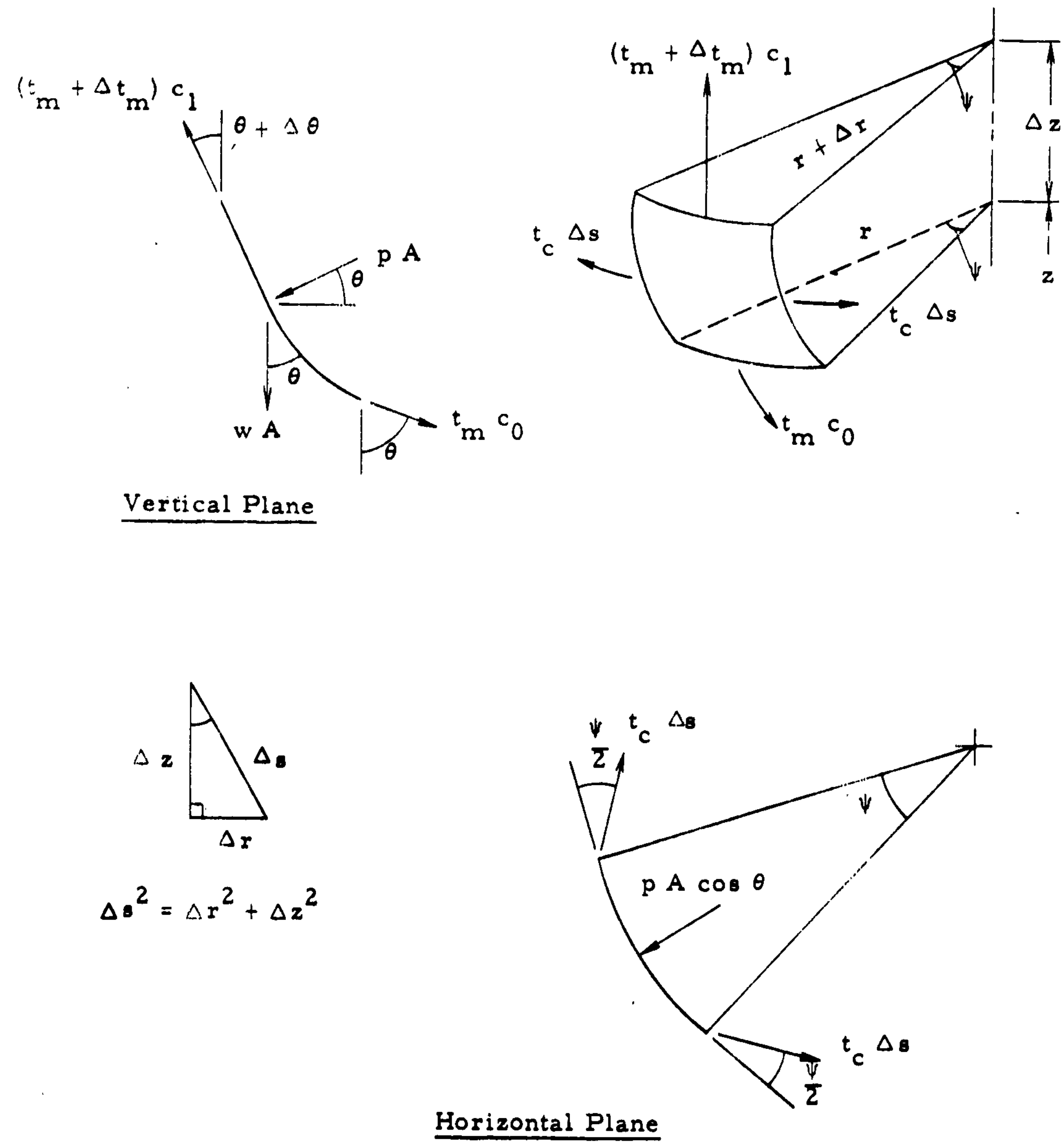


Figure 4. Sketch of Elemental Area

$$\Delta r\, t_m \cos\theta + r\,\Delta t_m \cos\theta - r\, t_m \sin\theta\,\Delta\theta - b\, r(z+a)\sin\theta\,\Delta s - r\, w\,\Delta s = 0.$$

The first three terms may be combined into $\Delta(r\, t_m \cos\theta)$. If the equation is divided by Δs, we have

$$\frac{\Delta(r\, t_m \cos\theta)}{\Delta s} - r\,[\; b(z+a)\sin\theta + w\;] = 0 \qquad (1)$$

Summing radial forces in the horizontal plane, we obtain -

$$(t_m + \Delta t_m)\, c_1 \sin(\theta + \Delta\theta) - t_m\, c_0 \sin\theta - 2t_c \sin(\psi/2)\,\Delta s + b(z+a)\cos\theta\, A = 0$$

or

$$\Delta r\, t_m \sin\theta + r\,\Delta t_m \sin\theta + r\, t_m \cos\theta\,\Delta\theta - t_c\,\Delta s + b\, r(z+a)\cos\theta\,\Delta s = 0$$

Then, as before,

$$\frac{\Delta(r\, t_m \sin\theta)}{\Delta s} - t_c + b\, r(z+a)\cos\theta = 0. \qquad (2)$$

In the limit, as $\Delta s \to 0$, Equations (1) and (2) become

$$\frac{d}{ds}(r\, t_m \cos\theta) - r\, w - b\, r(z+a)\sin\theta = 0$$

$$\frac{d}{ds}(r\, t_m \sin\theta) - t_c + b\, r(z+a)\cos\theta = 0.$$

Basically these equations define the stresses anywhere on the surface of a balloon with a given configuration.

Since

$$\sin\theta = dr/ds$$

and

$$\cos\theta = dz/ds,$$

another form of these equations is

$$\left.\begin{aligned} &\frac{d}{ds}(r\, t_m\, z') - r\, w - b\, r(z+a)\, r' = 0 \\ &\frac{d}{ds}(r\, t_m\, r') - t_c + b\, r(z+a)\, z' = 0 \end{aligned}\right\} \qquad (3)$$

where the prime (') denotes differentiation with respect to s. These are equations (6) and (7) as given in the University of Minnesota's report (Reference 1) vols. I and II.

Now

$$z' = dz/ds = dz/\sqrt{(dz)^2 + (dr)^2} = 1/\sqrt{1 + (dr/dz)^2}$$

$$r' = dr/ds = (dr/dz)\,(dz/ds) = (dr/dz)/\sqrt{1 + (dr/dz)^2}$$

$$\frac{d}{ds} = \frac{d}{dz}\frac{dz}{ds} = \left(1/\sqrt{1 + (dr/dz)^2}\right)\frac{d}{dz}.$$

If these are substituted in equation (3) above, we have

$$\frac{d}{dz}\left(\frac{r\, t_m}{\sqrt{1 + (r')^2}}\right) - r\, w\sqrt{1 + (r')^2} - br\,(z + a)\, r' = 0$$

$$\frac{d}{dz}\left(\frac{r\, t_m\, r'}{\sqrt{1 + (r')^2}}\right) - t_c\sqrt{1 + (r')^2} + br\,(z + a) = 0$$

where now

$$r' = dr/dz.$$

These are equations (1) and (2) in the same report.

E. Derivation of Shape Equations

Any of the foregoing equations define the shape of the balloon. The following development arrives at equations particularly suitable for computing purposes.

Returning to equation (3) and performing the indicated differentiation, we have

$$z''(r\, t_m) + z' \frac{d}{ds}(r\, t_m) - r\, w - br\,(z + a)\, r' = 0$$

$$r''(r\, t_m) + r' \frac{d}{ds}(r\, t_m) - t_c + br\,(z + a)\, z' = 0,$$

but, since $z'' = - r'\, \theta'$ and $r'' = z'\, \theta'$,

$$\left.\begin{aligned} - r'\, \theta'\,(r\, t_m) + z' \frac{d}{ds}(r\, t_m) - r\, w - br\,(z + a)\, r' &= 0 \\ z'\, \theta'\,(r\, t_m) + r' \frac{d}{ds}(r\, t_m) - t_c + br\,(z + a)\, z' &= 0. \end{aligned}\right\} \quad (4)$$

Eliminating the buoyancy term results in an explicit expression for the meridional material load:

$$\frac{d}{ds}(r\, t_m) = r\, w\, z' + r'\, t_c$$

Integrating produces the relationship

$$r\, t_m = \int_0^s (r\, w\, z' + r'\, t_c)\, ds + \text{constant.}$$

The constant had already been evaluated when it was noted that the total material load at the bottom ($s = 0$) of the balloon was

$$T_0 = 2\, \pi\, r\, t_m = L/\cos\, \theta_0$$

So

$$r\, t_m = L/2\, \pi \cos\, \theta_0 + \int_0^s (r\, w\, z' + r'\, t_c)\, ds. \quad (5)$$

Eliminating the term $\frac{d}{ds}(r\, t_m)$ from equation (4) results in

$$\theta'\,(r\, t_m) + r\, w\, r' - t_c\, z' + br\,(z + a) = 0$$

where use has been made of the fact that $(r')^2 + (z')^2 = 1$.

Rearranging,

$$- \theta' = \left[\, br\,(z + a) + r\, w\, r' - t_c\, z' \,\right] / r\, t_m. \quad (6)$$

This is the second equation, which, together with equation (5), is used to calculate shape. The terms a, L, θ_0, w, and t_c are independent variables. For given values of a and L and for given functions $w = w(s)$ and $t_c = t_c(s)$, the shape is uniquely determined by θ_0.

F. Non-Dimensional Equations and Derivation of Σ

Equations (5) and (6) are best non-dimensionalized by, first, factoring out b from the right hand side of (6) and dividing (5) by b. This produces

$$-\theta' = \left[r(z + a) + r(w/b)\ r' - (t_c/b)\ z' \right] / (r\ t_m/b) \tag{7}$$

where

$$(r\ t_m/b) = (L/b)/2\ \pi \cos \theta_0 + \int_0^s \left[r(w/b)\ z' + r'\ (t_c/b) \right] ds.$$

Second, let

$$\rho = r/(L/b)^{1/3}$$
$$\zeta = z/(L/b)^{1/3}$$
$$\alpha = a/(L/b)^{1/3}$$
$$\sigma = s/(L/b)^{1/3}$$
$$\overline{rt} = (r\ t_m/b)/(L/b)$$
$$\omega = (w/b)/(L/b)^{1/3}$$
$$\tau = (t_c/b)/(L/b)^{2/3}$$

and note that

$$\theta' = d\theta/ds = (d\theta/d\sigma)/(L/b)^{1/3}.$$

Substituting these in equation (7) above, results in the following non-dimensional form of the shape equations:

$$- d\theta / d\sigma = \left[\rho (\zeta + \alpha) + \rho \ \omega \ \rho' - \tau \ \zeta' \right] / \overline{rt}$$

where

$$\overline{rt} = 1/(2\pi \cos \theta_0) + \int_0^\sigma (\rho \ \omega \ \zeta' + \rho' \tau) \, d\sigma$$

Again, for a given α, θ_0, τ and ω the shape is determined.

The University of Minnesota presented results in Vol. IX (Reference 1) for the case $\alpha = \tau = 0$. In this case

$$- d\theta/d\sigma = (\zeta + \omega \rho') (\rho / \overline{rt}),$$

and

$$\overline{rt} = 1/(2\pi \cos \theta_0) + \int_0^\sigma \rho \ \omega \ \zeta' \, d\sigma$$

Here the shape is dependent on θ_0 and ω only. The University of Minnesota's results give a family of curves for a range of ω values with θ_0 chosen so that the balloon is flat on the top.

The term ω is a very useful balloon design parameter because the expression

$$\omega = (w/b) \ (b/L)^{1/3} = (1/L)^{1/3} \ (b)^{-2/3} \ w$$

is dependent on payload, altitude, and material weight. These are the three factors most likely to be known in the initial stages of a design. It has been reported (Reference 9) that the University of Minnesota defined

$$\Sigma = (2\pi/L)^{1/3} \ (V/G)^{2/3} \ w.$$

But

$$G = bV;$$

So

$$\Sigma = (2\pi/L)^{1/3} \ (b)^{-2/3} \ w = (2\pi)^{1/3} \ \omega .$$

The term Σ is well known and widely accepted; rather than introduce another parameter, it will be used herein.

Finally,

$$-d\theta/d\sigma = \left[\rho(\zeta + \alpha) + k\,\Sigma\,\rho\,\rho' - \tau\,\zeta'\right]/\overline{rt}$$

and (8)

$$\overline{rt} = 1/(2\pi\cos\theta_0) + \int_0^\sigma (k\,\Sigma\,\rho\,\zeta' + \tau\,\rho')\,d\sigma$$

where

$$k = (2\pi)^{-1/3}$$

It is clear the Σ will still be an important parameter when α and τ are non-zero.

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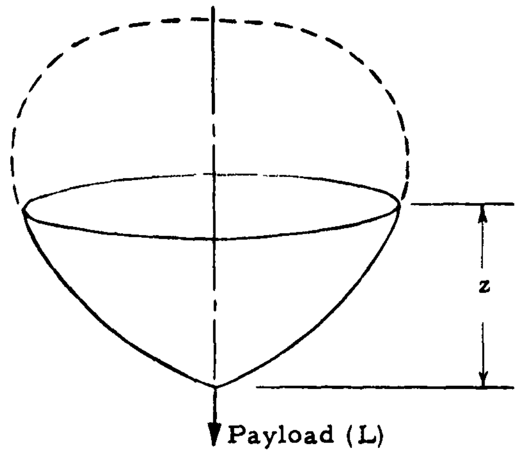
APPENDIX

DERIVATION OF THE DIFFERENTIAL EQUATIONS FOR THE BALANCE OF FORCES IN AN AXIALLY SYMMETRIC, THIN-FILM BALLOON

DERIVATION OF THE DIFFERENTIAL EQUATIONS FOR THE BALANCE OF FORCES IN AN AXIALLY SYMMETRIC, THIN-FILM BALLOON

1. VERTICAL FORCES

Consider the lower portion of the balloon as a free body, thus:



There are only four vertical forces to be considered: (1) the vertical components of meridional tension, considered positive upward; (2) the payload, considered positive downward; (3) the weight of the balloon below the height z, considered positive downward; and (4) the downward force due to the pressure acting on the horizontal projection of the balloon wall below height z, considered positive downward.

Therefore, when the pressure at height z is equal to $b(z + a)$, the weight per unit area of the balloon film is equal to w and the payload equal to L, (Figure A-1) the requirement that the sum of the vertical forces, at a height z, be equal to zero is satisfied by the following equation:

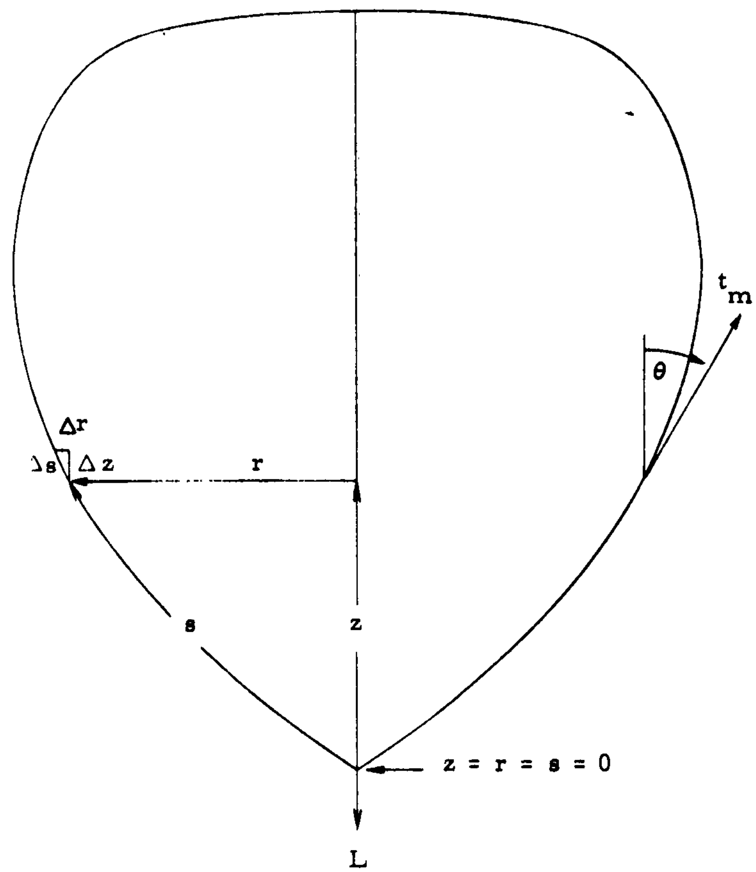


Figure A-1

$$\sum F_v = 2\pi r t_m \cos\theta - L - \int_0^s \int_0^{2\pi} b(z+a) \sin\theta \; r \, ds \, d\psi$$

$$- \int_0^s \int_0^{2\pi} rw \, ds \, d\psi = 0$$

which, upon integration with respect to ψ, yields

$$2\pi r t_m \cos\theta - L - \int_0^s 2\pi \, br\,(z+a) \sin\theta \, ds - \int_0^s 2\pi \, rw \, ds = 0.$$

Differentiating the latter equation with respect to s, and dividing by 2π we have

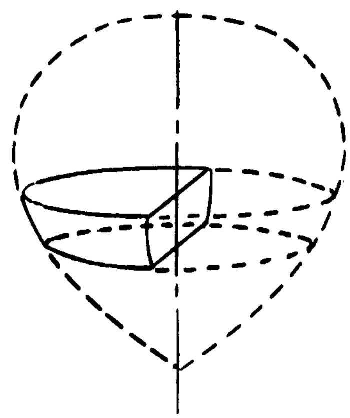
$$\frac{d}{ds}(r t_m \cos\theta) - br\,(z+a) \sin\theta - rw = 0$$

and, noting that $\cos\theta = dz/ds$ and $\sin\theta = dr/ds$, we can write

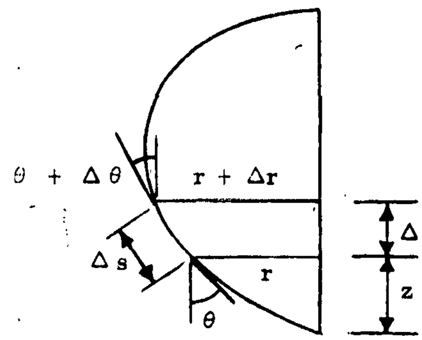
$$\frac{d}{ds}\left(r t_m \frac{dz}{ds}\right) - br\,(z+a)\frac{dr}{ds} - rw = 0.$$

II. HORIZONTAL FORCES

Consider as a free body a portion of the balloon, as defined in the following sketch



This is a semi-circular element bounded by two horizontal planes (normal to the axis of symmetry) and by a vertical plane (containing the axis of symmetry). The location and dimensions of this free body are defined in the following sketch:



An enlarged view of the free body is shown in Figure A-2. The material stresses are also indicated. Since only horizontal loads are to be considered, the weight of the material is omitted. In addition to the material stresses, there is a pressure normal to the surface of magnitude b(z + a). Horizontal forces in the direction of t_c will be considered, and are discussed individually in the following:

A. Circumferential Stress

Assuming the variation of t_c is linear for small Δs then

$$F_c = 2\left[\frac{t_c + (t_c + \Delta t_c)}{2}\right]\Delta s = (2\,t_c + \Delta t_c)\,\Delta s.$$

If $\Delta t_c << 1$ and $\Delta s << 1$ and we omit second order terms, then

$$F_c = 2\,t_c\,\Delta s.$$

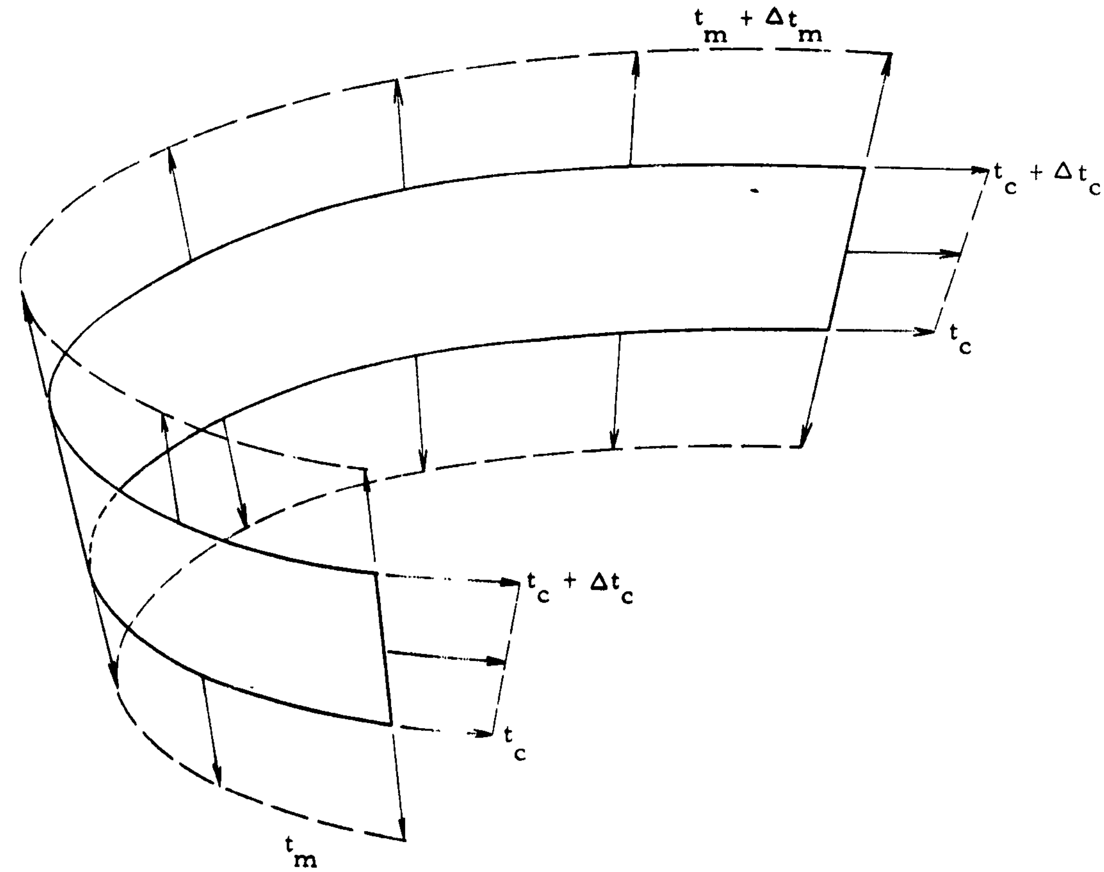


Figure A-2

B. Meridional Stress

In the z-plane the horizontal component of meridional stress is $t_m \sin\theta$ which, when integrated for its resultant in the t_c direction, is

$$2\, r\, t_m \sin\theta .$$

In the $(z + \Delta z)$-plane the horizontal component of meridional stress is

$$(t_m + \Delta t_m) \sin(\theta + \Delta\theta).$$

If this is integrated for its resultant in the t_c direction, we have

$$2(r + \Delta r)\,(t_m + \Delta t_m) \sin(\theta + \Delta\theta).$$

If $\Delta r << 1$, $\Delta t_m << 1$ and $\Delta\theta << 1$ and we omit second order terms, then

$$F_m = 2r(t_m \sin\theta + t_m \cos\theta\ \Delta\theta + \sin\theta\ \Delta t_m)$$
$$+ 2\, t_m \sin\theta\ \Delta r.$$

Note that

$$\Delta(r\, t_m \sin\theta) = r\, t_m \cos\theta\ \Delta\theta + r \sin\theta\ \Delta t_m + t_m \sin\theta\ \Delta r.$$

Then

$$F_m = 2\, r\, t_m \sin\theta + 2\,\Delta(r\, t_m \sin\theta).$$

C. Pressure Force

From elementary hydrostatics the resultant pressure force will be equal to the area of the balloon surface projected into the vertical plane times the pressure at the centroid of that area. If we assume that the variation of r is linear for small Δz, then the projected area is

$$2\left[\frac{r + (r + \Delta r)}{2}\right]\Delta z.$$

The centroid of this area, correct to first order, is at $z + \Delta z/2$. The pressure then is

$$b(z + \frac{\Delta z}{2} + a).$$

If $\Delta r << 1$ and $\Delta z << 1$ and we omit second order terms the resultant pressure force is

$$F_p = 2\ r\ b(z + a)\ \Delta z.$$

Summing the three forces derived above in the direction of t_c with consideration for their direction as well as magnitude we have

$$\sum F_h = t_c\ \Delta s - \Delta(r\ t_m\ \sin\theta) + r\ b(z + a)\ \Delta z = 0.$$

Dividing by Δs, taking the limit as $\Delta s \rightarrow 0$ and noting that $\sin\theta = dr/ds$ we finally have

$$\frac{d}{ds}(r\ t_m \frac{dr}{ds}) + r\ b(z + a)\frac{dz}{ds} - t_c = 0.$$